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Quasi-point separation of variables for the Henon–Heiles system and a system with a quartic potential

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Abstract. We examine the problem of integrability of two-dimensional Hamiltonian systems by means of separation of variables. A systematic approach to the construction of the special non-pure coordinate separation of variables for certain natural two-dimensional Hamiltonians is proposed.

1. Introduction

In this paper we study quasi-point separation of variables for certain natural Hamiltonians

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x, y) \tag{1.1}$$

of two degrees of freedom.

The classical separability theory [10, 4] is concerned with orthogonal point transformations $x = \Phi_1(u, v)$ and $y = \Phi_2(u, v)$ for which the corresponding Hamilton–Jacobi equation expressed in terms of (u, v)-variables can be solved by separation of variables. The list of coordinate systems in-plane that provide point separation of variables for systems with the Hamiltonian (1.1) are well known. Even an effective criterion of separability for a given potential $V(x, y)$ has been formulated [5, 10]. In this paper we consider a non-pure point transformation related to a Cartesian system of coordinates.

Very little is known about the general canonical transformations $x_k = \Phi_k(u_j, p_{u_j})$ and $p_k = \Psi_k(u_j, p_{u_j})$, $k = 1, 2$, which separate the Hamiltonian (1.1). Recently, a special type of non-point transformations $x = \varphi(H, C) \cdot \Phi(u, v)$ has been introduced for the third integrable case of the Henon–Heiles system [7, 3] and for a system with a quartic potential [8]. Here $\varphi(H, C)$ is a certain function of the Hamiltonian H and of the second integral of motion C. On the orbit $\mathcal{O}$ $(H = \alpha_1, C = \alpha_2)$ this transformation becomes a point transformation and, therefore, we shall call it a quasi-point transformation. These transformations have been found either in the context of the Painlevé expansion [7, 8] or have been derived from the Miura transformation for the systems related to stationary flows of soliton equations [3, 2]; however, no general principle for constructing general (i.e. non-point) canonically separable potentials is known. This is the reason why in this paper we address this question directly by starting with general separated equations and by trying to find transformations which lead to natural Hamiltonians (1.1) of two degrees of freedom. We do not find any essentially new potentials. We present in detail our construction of separating variables and list the main results. We explain the connection of quasi-point transformations with supersymmetric quantum mechanics.

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2. Quasi-point canonical transformation

Let us begin with a two-dimensional system expressed in terms of canonical variables of separation (u, p_u, v, p_v) with the rectangular separated equations of the general form [4, 10]

$$\Delta_1(u, p_u) = f(u)p_u^2 + V_1(u) \qquad \Delta_2(v, p_v) = g(v)p_v^2 + V_2(v)\,. \tag{2.1}$$

The functions Δ_j Poisson commute $\{\Delta_1, \Delta_2\} = 0$ with respect to the standard Poisson brackets. Notice that we could, by the using a point transformation, reduce these equations to equations with $f(u) = g(v) = 1$, but the form of (2.1) is more suitable for further calculations.

As commuting integrals of motion we choose

$$\begin{aligned} &H = \Delta_1 + \Delta_2 = f(u)p_u^2 + g(v)p_v^2 + V_+(u, v) \\ &C = a\,\Delta_1 - b\,\Delta_2 = a\,f(u)p_u^2 - b\,g(v)p_v^2 + V_-(u, v) \qquad a, b \in \mathbb{R} \end{aligned} \tag{2.2}$$

which clearly define $V_\pm(u, v)$. These integrals are second-order polynomials in momenta. A condition of separability for the potentials $V_\pm$ is

$$\frac{\partial^2}{\partial u\,\partial v} V_\pm(u, v) = 0\,. \tag{2.3}$$

Let us denote a canonical transformation to the Cartesian variables (x, y) as

$$x = \Phi_1(u, p_u, v, p_v) \qquad \text{and} \qquad y = \Phi_2(u, p_u, v, p_v) \tag{2.4}$$

and let us require that after the transformation (2.4) the Hamiltonian (2.2) takes the natural Hamiltonian form (1.1). Then

$$p_x = \{H, x\} \qquad \text{and} \qquad p_y = \{H, y\}$$

and, after substituting these momenta into the canonical Poisson brackets, we find a system of equations for the transformations (2.4), which reads as

$$\begin{aligned} &\{\Phi_j(u, p_u, v, p_v), \Phi_k(u, p_u, v, p_v)\} = 0 \\ &\{\{H, \Phi_j(u, p_u, v, p_v)\}, \{H, \Phi_k(u, p_u, v, p_v)\}\} = 0 \qquad (j, k = 1, 2) \\ &\{\{H, \Phi_j(u, p_u, v, p_v)\}, \Phi_k(u, p_u, v, p_v)\} = \delta_{jk}\,. \end{aligned} \tag{2.5}$$

Since, we cannot solve this system of equations in the whole generality, we have to make a certain simplifying ansatz for the functions Φ_k in order to obtain certain particular solutions of (2.5).

In the class of point transformations $\widetilde{x} = \widetilde{\Phi}_1(u, v)$ and $\widetilde{y} = \widetilde{\Phi}_2(u, v)$, the general solution of (2.5) has the form

$$\widetilde{x} = \alpha u + \beta v + \lambda_1 \qquad \widetilde{y} = \gamma u + \delta v + \lambda_2 \tag{2.6}$$

which is a superposition of rotation and translation transformations. Since the functions $f(u)$, $g(v)$ and $V_\pm(u, v)$ are arbitrary we can always rewrite the solution (2.6) in the following non-symmetric form:

$$\widetilde{x} = \widetilde{\Phi}_1(u, v) = u - v \qquad \widetilde{y} = \widetilde{\Phi}_2(u, v) = u + v + 2\gamma \tag{2.7}$$

which will be used below. The polynomial order of integrals of motion (2.2) remains unchanged under (2.7).

In order to obtain a non-point transformation as a solution of (2.5) we shall introduce a particular ansatz for $x = \Phi_1(u, p_u, v, p_v)$, which is suggested by the results of [7, 8]. To explain the structure of this ansatz we begin with a simple example, which will be present in detail below. Let us take

$$\Delta_j(\xi) = \xi^n p_\xi^2 + V_j(\xi) \qquad \xi = u, v$$
$$x^2 = \Delta_1(u)u^{-m} \qquad u^m = \Delta_1 x^{-2} \tag{2.8}$$

and require that the Hamiltonian (2.2) takes the natural form (1.1), when expressed through the new variables (x, p_x). Then, we obtain

$$\frac{p_x}{x} = -mu^{n-1}p_u$$

and by applying the Poisson brackets (2.5) we get

$$\left\{\frac{p_x}{x}, x^2\right\} = 2 = \{-mu^{n-1}p_u, u^{n-m}p_u^2 + u^{-m}V_1(u)\}. \tag{2.9}$$

The kinetic part of this equation leads to a restriction for the kinetic part of the Hamiltonian: $n = 2 - m$. The potential part of (2.9) gives

$$u\frac{\partial V_1(u)}{\partial u} - mV_1(u) = -2m^{-1}u^{2m}$$

which has the solution

$$V_1 = -2m^{-2}u^{2m} + \alpha u^m = -2m^{-2}\Delta_1^2 x^{-4} + \alpha\Delta_1 x^2 \qquad \alpha \in \mathbb{R}. \tag{2.10}$$

According to the definitions (1.1) and (2.8) the momenta p_u is

$$p_u = -\frac{\Delta_1 p_x x^{-3}}{mu}$$

and, after substituting u, p_u and $V_1(u)$, the separated equation Δ_1 (2.8) becomes

$$\Delta_1(x, p_x) = \tfrac{1}{2}p_x^2 - \tfrac{1}{2}m^2(x^4 - \alpha x^2) \qquad \alpha \in \mathbb{R}.$$

The second pair of variables (y, p_y) follows from the second separated equation $\Delta_2(v)$ and the Hamiltonian (2.2) transforms to the new Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \alpha_1 x^4 + \alpha_2 y^4 + \alpha_3 x^2 + \alpha_4 y^2 \qquad \alpha_k \in \mathbb{R}$$

under the non-point canonical transformation (2.8).

Next we consider a symmetric form of this transformation related to the symmetric form of the second integral of motion (2.2). Let us take

$$P_n(x) = \sum_{k=0}^{n} a_k x^k = C\phi(u - v) \qquad a_k \in \mathbb{R} \tag{2.11}$$

where $P_n(x)$ is an nth-order polynomial, C is an integral of motion (2.2) and $\phi(u - v)$ is a function of one variable $z = u - v$. On the orbit $\mathcal{O}$ ($C = c =$ constant) the transformation (2.11) becomes a point transformation and, therefore, we shall call this transformation a quasi-point transformation.

After substituting the ansatz (2.11) into the equation

$$\{\{H(u, p_u, v, p_v), \Phi_1(u, p_u, v, p_v)\}, \Phi_1(u, p_u, v, p_v)\} = 1 \tag{2.12}$$

we consider terms at independent powers of momenta. It appears that a solution of the corresponding equation exists only if $P_n(x) = x^2$ (up to a point transformation (2.6)) and we obtain one equation for the function $\phi(z)$

$$2\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} = \phi(z)\frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} \qquad z = u - v \tag{2.13}$$

and a system of equations for the functions $f(u)$ and $g(v)$

$$\begin{aligned} &f(u) - bg(v) - b(u-v)\frac{\mathrm{d}g(v)}{\mathrm{d}v} = 0 \\ &-af(u) + bg(v) + a(u-v)\frac{\mathrm{d}f(u)}{\mathrm{d}u} = 0\,. \end{aligned} \tag{2.14}$$

The general solutions for (2.13) and (2.14) are

$$\begin{aligned} &\phi(z) = z^{-1} = \widetilde{\Phi}_1^{-1}(u,v) \\ &a = b = \alpha_1 \qquad f(u) = \alpha_2 u + \alpha_3 \\ &g(v) = \alpha_2 v + \alpha_3 \qquad \alpha_k \in \mathbb{R}\,. \end{aligned} \tag{2.15}$$

Further, we obtain that

$$x^2 = C\widetilde{\Phi}_1^{-1}(u,v)$$

where $\widetilde{\Phi}_1^{-1}(u,v)$ is a point transformation (2.7) and, therefore, we can consider (2.11) as a natural generalization of a pure point transformation, which can be applied to the other three types of coordinate-separated equations in the plane. For instance, for the parabolic system of coordinates we have to substitute

$$C = \frac{v^2\Delta_1 - u^2\Delta_2}{u^2+v^2} \qquad \widetilde{\Phi}_1(u,v) = u^2 - v^2$$

into the ansatz (2.11).

From the remaining terms of (2.12) (by taking into account (2.13) and (2.14)) we obtain that the potential $V_-(u,v)$ obeys the following equation:

$$\left((\alpha_2 u + \alpha_3)\frac{\partial V_-}{\partial u} - (\alpha_2 v + \alpha_3)\frac{\partial V_-}{\partial v}\right) - \frac{(\alpha_2(u+v) + 2\alpha_3)}{\alpha_1(u-v)} V_- = -\frac{2(u-v)^3}{\alpha_1(u-v)}\,. \tag{2.16}$$

For a separable potential $\partial^2 V_-(u,v)/\partial u\partial v = 0$ this equation has a unique solution

$$V_{1,2}(\xi) = \frac{2/\alpha_1\xi^3 + \beta_2\xi^2 + \beta_1\xi + \beta_0}{\alpha_2\xi + \alpha_3} \qquad \xi = u, v \tag{2.17}$$

where β_k are arbitrary constants.

Next we have to determine the second variable $y = \Phi_2(u, p_u, v, p_v)$ by using equations (2.5). Again we are not able to find a general solution for Φ_2 and we shall use the particular ansatz of [7, 8],

$$Q_n(y) = \sum_{k=0}^{n} b_k y^k = \psi_1(u+v) + \psi_2(x, p_x) \qquad p_y = \{H, y\} \tag{2.18}$$

where $Q_n(y)$ is a polynomial of order n and ψ_k are as yet unspecified functions.

After substituting (2.18) into the equations

$$\{x, Q_n(y)\} = 0 \qquad \text{and} \qquad \{p_x, Q_n(y)\} = 0$$

(recall that $p_x = \{H, x\}$) we obtain

$$Q_n(y) = \gamma_1\left[u + v + \gamma_2 - \frac{\alpha_1}{2}\left(\frac{p_x}{x}\right)^2 + \frac{\alpha_2 x^2}{2}\right]. \tag{2.19}$$

This ansatz is a superposition of a point transformation $\widetilde{\Phi}_2(u,v)$ (equation (2.7)) with a term depending on the first variables (x, p_x).

By using the remaining equation $\{p_y, y\} = 1$ we obtain (up to a point transformations (2.6)) that

$$Q_n(y) = y^2 \qquad \gamma_1 = -\frac{1}{\alpha_2} \tag{2.20}$$

and yet another differential equation of the second order for the potential $V_+(u, v)$. We present here a partial form of this equation with $\alpha_2 = 0$

$$\frac{\partial^2 V_+}{\partial z^2} = z\frac{\partial V_+}{\partial z} \qquad z = u - v\,.$$

This equation for V_+ is equivalent to the equation for V_- (equation (2.16)) provided that the constants $\alpha_1,\ \alpha_2,\ \beta_2$ and γ_2 satisfy

$$\gamma_2 + \alpha_1\beta_2 + 2\alpha_3 = 0\,. \tag{2.21}$$

Let us summarize these considerations as

Proposition 1. A quasi-point transformation of the form

$$P_n(x) = \sum_{k=1}^{n} a_k x^k = \frac{C}{u-v}$$
$$Q_m(y) = \sum_{k=1}^{m} b_k y^k = \psi_1(u+v) + \psi_2(x, p_x) \qquad a_k, b_k \in \mathbb{R}$$

extended to a canonical transformation, transforms the Hamiltonian

$$H = f(u)p_u^2 + g(v)p_v^2 + V_1(u) + V_2(v)$$

into the natural Hamiltonian form

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x, y)$$

if and only if

$$\begin{aligned}
x^2 &= \frac{C}{u-v}\\
\frac{p_x}{x} &= -\frac{(\alpha_2 u + \alpha_3)p_u - (\alpha_2 v + \alpha_3)p_v}{u-v}\\
y^2 &:= -\frac{1}{\alpha_2}\left(u + v - \alpha_1\beta_2 - 2a_3 - \frac{\alpha_1}{2}\left(\frac{p_x}{x}\right)^2 + \frac{\alpha_2 x^2}{2}\right)\\
yp_y &= \frac{1}{\alpha_2}\Bigg((\alpha_2 u + \alpha_3)p_u - (\alpha_2 v + \alpha_3)p_v\\
&\qquad + \frac{p_x}{2x}\left(\frac{\alpha_1 p_x^2}{x^2} + \frac{6\alpha_3}{\alpha_2} - 6\alpha_2 y^2 + \alpha_1\beta_2 - \alpha_2 x^2\right)\Bigg)
\end{aligned} \tag{2.22}$$

and the potentials

$$V_k(\xi) = \frac{2/\alpha_1\xi^3 + \beta_2\xi^2 + \beta_1\xi + \beta_0}{\alpha_2\xi + \alpha_3} \qquad k = 1 \quad \xi = u \quad \text{or} \quad k = 2 \quad \xi = v\,.$$

Constants $\alpha_j, \beta_j,\ j = 1, 2, 3$ are the six free parameters of this transformation.

An inverse transformation to (2.22) reads

$$\begin{aligned} \xi &= \pm\frac{C}{2x^2} + \frac{\alpha_1 p_x^2}{4x^2} - \frac{\alpha_2}{4}(2y^2+x^2) - \gamma_2 \qquad \xi = u, v \\ p_\xi &= \frac{1}{2(\alpha_2\xi+\alpha_3)}\left(\frac{p_x C}{x^3} - \alpha_2 y p_y - \frac{p_x}{2x}\left(\frac{\alpha_1 p_x^2}{x^2} + \frac{6\alpha_3}{\alpha_2} - 6\alpha_2 y^2 + \alpha_1\beta_2 - \alpha_2 x^2\right)\right) \end{aligned} \tag{2.23}$$

where the integral C is an as yet unspecified function of the new variables (x, p_x, y, p_y). We have to substitute this inverse transformation into the definition of the integral of motion C (equation (2.2)) and solve the resulting equation. It has the polynomial form

$$aC^4 + bC^2 + c = 0 \tag{2.24}$$

with the coefficients a, b and c depending on the variables (x, p_x, y, p_y). There are two solutions $C_{1,2}^2$, which are related by the change of variables $y \to -y$, $p_y \to -p_y$, and they are polynomial expressions in the new variables. Notice that the second integral of motion C (equation (2.2)), which is polynomial in (u, p_u, v, p_v) becomes an algebraic function of (x, p_x, y, p_y).

After substitution of $C(x, p_x, y, p_y)$ into (2.23) we get an inverse transformation.

Proposition 2. A Hamiltonian

$$H = f(u)p_u^2 + g(v)p_v^2 + V_1(u) + V_2(v)$$

with the potentials given by (2.17) transforms to either of two natural Hamiltonians

$$H_1 = \tfrac{1}{2}(p_x^2+p_y^2) - \frac{\alpha_2}{2\alpha_1}(x^4+6x^2y^3+8y^4) + \tfrac{1}{2}\beta_2(x^2+4y^2) + \frac{2\beta_0}{\alpha_2^2 y^2} + 2\alpha_2\beta_1 \tag{2.25}$$

and

$$H_2 = \tfrac{1}{2}(p_x^2+p_y^2) - \frac{\alpha_2}{2\alpha_1}(x^4+6x^2y^3+y^4) \tag{2.26}$$

under the quasi-point transformation (2.22) and under the point transformation, respectively.

For brevity, we put $\alpha_3 = 0$ in H_1 (equation (2.25)), since the introduction of $\alpha_3 \neq 0$ corresponds to the shift of the variable $y \to y + \alpha_3/4\alpha_2$ [9]. In H_2 (equation (2.26)) we presented only the highest polynomial terms.

Systems with Hamiltonians H_1 and H_2 are associated with restricted flows of some PDEs [3, 2]. In classical mechanics these systems have common separated equations. Second integrals of motion C^2 derived from the equation (2.24) are well known and can be found in [8, 9].

By rescaling constants a_j, β_j and by taking the limits $\alpha_2 \to 0$ and $\alpha_3 \to 1$, the Hamiltonians (2.25) and (2.26) are transformed into the following Hamiltonians for the Henon–Heiles system [3, 7]:

$$\begin{aligned} H_1 &= \tfrac{1}{2}(p_x^2+p_y^2) + \frac{2}{\alpha_1\alpha_3}y(3x^2+16y^2) - \beta_2(x^2+16y^2) \\ &\quad + 2(\alpha_1\beta_2 - 2\beta_1)y + 2\beta_0 + \alpha_1\beta_2\beta_1 \\ H_2 &= \tfrac{1}{2}(p_x^2+p_y^2) + ay(x^2+2y^2)\,. \end{aligned} \tag{2.27}$$

A complete account of this limit procedure can be found in [9].

It is known that the Hamiltonian (2.25) has integrable extensions

$$H = H_{1,2} + \frac{m}{x^2} + \frac{l}{y^2} + \frac{n}{x^6} + ey$$

where either $e = 0$ or $n = m = 0$ [8]. We can include the terms μx^{-2} and $l y^{-2}$ in our proposed scheme. Let us consider the following infinite-dimensional representation of $sl(2)$ defined in the Cartan–Weil basis:

$$s_3 = \frac{xp}{2} \qquad s_+ = \frac{x^2}{2} \qquad s_- = -\frac{p^2}{2} \qquad \{p, q\} = 1\,.$$

The mapping

$$\begin{aligned} &s_3 \to s_3' = s_3 \qquad s_+ \to s_+' = s_+ \\ &s_- \to s_-' = s_- + f s_+^{-1} = \frac{p^2}{2} + \frac{2f}{x^2} \qquad f \in \mathbb{R} \end{aligned} \tag{2.28}$$

is an outer automorphism of the space of infinite-dimensional representations of $sl(2)$. For the Henon–Heiles system and for the system with a quartic potential, the phase space can be identified completely or partially with the coadjoint orbits in $sl(2)^*$ as (2.28). Hence, all the presented results can be carried over on the systems with shifted squared momenta. The corresponding deformation of the separated equation has been described in [8].

3. Quasi-point transformations and SUSY quantum mechanics

Next, we present the interesting relations of the quasi-point canonical transformation with the supersymmetrical quantum mechanics, which represents in a concise algebraic form the spectral equivalence between different Hamiltonian quantum systems realized by the Darboux transformation [1]. Further, for brevity, we fix the value of parameters $\beta_k = 0$, $\alpha_2 = 1$, $\alpha_1 = 2$.

Variables of separation u, v for the Hamiltonian H_2 can be defined as the roots of quadratic equation

$$\xi^2 - y\xi + \frac{y^2 - x^2}{4} = 0\,. \tag{3.1}$$

The quasi-point transformations (2.23) can be presented in a similar form. Variables u, v (equation (2.23)) are roots of the quadratic equation

$$\xi^2 - \frac{(q_+ + q_-)}{x^2}\xi + \frac{(q_+ - q_-)^2}{4x^2} = 0\,. \tag{3.2}$$

Here, we introduced functions $q_\pm$ and f with the following properties:

$$\{H, q_\pm\} = \pm f\, q_\pm \qquad C^2 = 4 q_+ q_-\,. \tag{3.3}$$

This algebra (3.3) is the classical limit of the two-dimensional quantum SUSY algebra [1]. For the system with a quartic potential, functions $q_\pm$ and f are given by

$$\begin{aligned} &q_+ = \tfrac{1}{2} p_x^2 - \tfrac{1}{4} x^2 (2y^2 + x^2) + x\big(y p_x - \tfrac{1}{2} p_y x\big) \\ &q_-(x, p_x, y, p_y) = q_+(x, p_x, -y, -p_y) \\ &f = 2y \end{aligned} \tag{3.4}$$

and for the Henon–Heiles system

$$\begin{aligned} &q_+ = \tfrac{1}{2} p_x^2 + 2x^2 y + \mathrm{i} x \sqrt{2(2p_x y - x p_y) p_x - (8y^2 + x^2) x^2} \\ &q_-(x, p_x, y, p_y) = q_+^*(x, p_x, y, p_y) \\ &f = \mathrm{i} \frac{4 p_x y - x p_y}{\sqrt{2(2p_x y - x p_y) p_x - (8y^2 + x^2) x^2}} \end{aligned} \tag{3.5}$$

where i is $\sqrt{-1}$ and q_+^* denotes the complex conjugate of the function q_+. For the arbitrary values of the parameters α_k and β_k these functions are derived from equations (2.23) and (3.2).

For the system with quartic potential functions $q_\pm$ and f, equations (3.4) have been introduced in [6] by considering a linearization of the corresponding Hamiltonian flow on certain constraint submanifolds.

It would be interesting to apply equation (3.2) and its quantum counterpart for separating variables in quantum mechanics.

Motivated by the relation between the classical limit of SUSY quantum mechanics and quasi-point separation of variables we present the classical SUSY algebra (3.3) for some other two-dimensional natural Hamiltonian systems. For the systems of [11] and for the Holt-like systems the classical limit of SUSY algebra (3.3) is equal to

$$H_W = \frac{p_x^2+p_y^2}{2} + \left(\frac{a_1^2}{2}+\alpha_2\alpha_3\right)x^{2/3} - \frac{9}{16}\alpha_1^2\alpha_2 x^{-2/3}y - \alpha_3 y$$

$$q_+ = -\frac{p_x^2}{2} + \frac{9}{16}\left(\alpha_1^2\alpha_2 x^{-2/3}y - \frac{\alpha_2}{\alpha_1}\right) + \left(\frac{\alpha_1^2}{2}+\alpha_2\alpha_3\right)x^{2/3} + \mathrm{i}\left(\alpha_1 x^{1/3}p_x + \frac{\alpha_2}{3\alpha_1}p_y\right)$$

$$f = -\tfrac{2}{3}\mathrm{i}\alpha_1 x^{-2/3}$$

and

$$H_H = \frac{p_x^2+p_y^2}{2} + y^{-2/3}\left(\tfrac{9}{2}y^2+x^2\right)$$

$$q_+ = p_x^2 + 2y^{-2/3}x^2 + \mathrm{i}\sqrt{2(p_xp_y+6xy^{1/3})^2-(2x^2y^{-2/3})^2}$$

$$f = \frac{4\mathrm{i}x^2p_y}{3y\sqrt{2(y^{2/3}p_xp_y+6yx)^2-4x^4}}$$

where $q_-(x,p_x,y,p_y) = q_+^*(x,p_x,y,p_y)$.

Notice, that the construction of isospectral two-dimensional Hamiltonians in supersymmetrical quantum mechanics is closely connected with another problem, namely with a search for the second integral of motion for the quantum integrable systems [1]. Here we present a new characterization of this problem.

Proposition 3. Let us start with the classical SUSY algebra (3.3) for a two-dimensional integrable system defined by four functions H, f and $q^\pm$. If the following equation for Δh can be solved:

$$\{\Delta h, \{q_+, q_-\}\} = \{f, q_+\}q_- + \{f, q_-\}q_+ \tag{3.6}$$

then the pair of mutually commuting integrals of motion

$$\widetilde{H} = H + \Delta h \qquad \widetilde{C} = \{q^+, q^-\} \qquad \{\widetilde{H}, \widetilde{C}\} = 0 \tag{3.7}$$

defines a new two-dimensional integrable system.

The evolution on a $2n$-dimensional symplectic manifold is called completely integrable if there exist n functions $I_1,\ldots,I_n$, which are independent integrals in the involution

$$\{I_i(x,p), I_j(x,p)\} = 0 \qquad i,j = 1,\ldots,n\,.$$

The initial integrals of motion H and C are functionally independent functions and, therefore, the new integrals $\widetilde{H}$ and $\widetilde{C}$ are independent. For the proof of involution of integrals we need the Jacobi identity for the Poisson brackets. First, assuming (3.3) holds we have

$$\{H,\widetilde{C}\} = \{H,\{q^+,q^-\}\} = \{q^+,\{H,q^-\}\} - \{q^-,\{H,q^+\}\}$$
$$= q^-\{f,q^+\} + q^+\{f,q^-\}\,.$$

Next, using (3.6) we get (3.7)

$$\{H+\Delta h,\widetilde{C}\} = \{\widetilde{H},\widetilde{C}\} = 0\,.$$

So, we have two independent integrals of motion in involution, which define a new integrable system.

As an example, for the system with a quartic potential, equation (3.6) can be integrated and it yields

$$\begin{aligned} H_1 &= \frac{p_x^2+p_y^2}{2} - \tfrac{1}{4}(x^4+6x^2y^2+8y^4) \\ \Delta h &= -\tfrac{1}{8}x^4 \\ \widetilde{H} &= H_1+\Delta h = \frac{p_x^2+p_y^2}{2} - \tfrac{1}{8}(x^4+12x^2y^2+16y^4) \\ \widetilde{C} &= \{q_+,q_-\} = 16\big(2(p_yx-yp_x)p_x-(2y^2-x^2)yx^2\big)\,. \end{aligned} \tag{3.8}$$

A new system with integrals of motion $\widetilde{H},\widetilde{C}$ is separable in the parabolic coordinates, which are defined as roots of the quadratic equation $\xi^2+2y\xi-x^2=0$.

4. Conclusions

In the previous paragraphs we have examined the six-parameter quasi-point canonical transformation leading to separation of variables for the Henon–Heiles system and a system with a quartic potential. We have proved that this transformation is strictly connected to the rectangular separated equations and that it cannot be generalized. In addition, we have found some relations of these variables of separation with the two-dimensional supersymmetric quantum mechanics.

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